Further mathematics
Higher level
Paper 2

Thursday 11 May 2017 (morning)

2 hours 30 minutes

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[150 marks]**.

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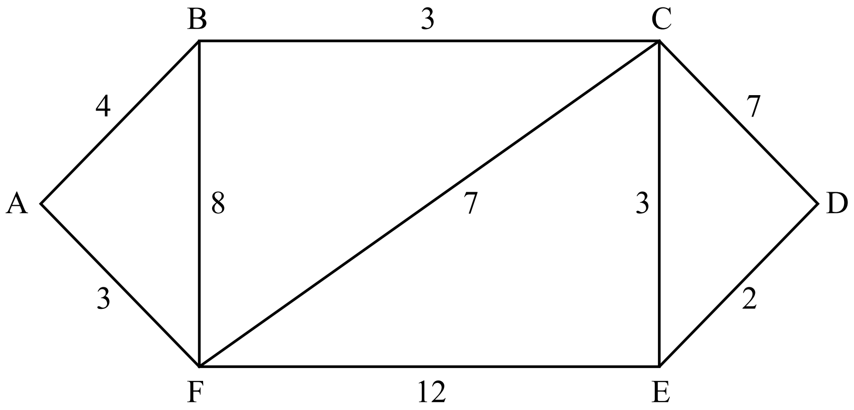
Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 11]



The diagram shows the graph G with the weights of the edges marked.

(a) (i) State what features of the graph enable you to state that G contains an Eulerian trail but no Eulerian circuit.

(ii) Write down an Eulerian trail. [4]

(b) Use Dijkstra's algorithm to find the path of minimum total weight joining A to D, stating this total weight. Your solution should show clearly that this algorithm has been used. [7]

2. [Maximum mark: 15]

(a) The group $\{G, *\}$ has a subgroup $\{H, *\}$. The relation R is defined such that for $x, y \in G, xRy$ if and only if $x^{-1} * y \in H$. Show that R is an equivalence relation. [8]

(b) Consider the special case in which $G = \{1, 3, 4, 9, 10, 12\}$, $H = \{1, 12\}$ and $*$ denotes multiplication modulo 13.

(i) Show that $3R10$.

(ii) Determine the three equivalence classes. [7]

3. [Maximum mark: 15]

(a) The sequence $\{u_n\}$ satisfies the second-degree recurrence relation

$$u_{n+2} = u_{n+1} + 6u_n,\ n \in \mathbb{Z}^+.$$

(i) Write down the auxiliary equation.

(ii) Given that $u_1 = 12$, $u_2 = 6$, show that

$$u_n = 2 \times 3^n - 3 \times (-2)^n.$$

(iii) Determine the value of $\lim_{n \to \infty} \frac{u_n + u_{n-1}}{u_n - u_{n-1}}$. [11]

(b) Another sequence $\{v_n\}$ is such that

$$v_n = u_{2n},\ n \in \mathbb{Z}^+.$$

Determine the second-degree recurrence relation satisfied by $\{v_n\}$. [4]

4. [Maximum mark: 19]

(a) Consider the matrix

$$A = \begin{bmatrix} \lambda & 3 & 2 \\ 2 & 4 & \lambda \\ 3 & 7 & 3 \end{bmatrix}.$$

(i) Find an expression for $\det(A)$ in terms of λ, simplifying your answer.

(ii) Hence show that A is singular when $\lambda = 1$ and find the other value of λ for which A is singular. [5]

Suppose now that $\lambda = 1$ so consider the matrix

$$B = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 4 & 1 \\ 3 & 7 & 3 \end{bmatrix}.$$

(b) (i) Explain how it can be seen immediately that B is singular without calculating its determinant.

(ii) Determine the null space of B.

(iii) Explain briefly how your results verify the rank-nullity theorem. [7]

(c) Prove, using mathematical induction, that

$$B^n = 8^{n-2}B^2 \text{ for } n \in \mathbb{Z}^+, n \geq 3.$$ [7]

5. [Maximum mark: 24]

(a) (i) By considering integration as the reverse of differentiation, show that for $0 \leq x < \frac{\pi}{2}$

$$\int \sec x \, \mathrm{d}x = \ln(\sec x + \tan x) + C .$$

(ii) Hence, using integration by parts, show that

$$\int \sec^3 x \, \mathrm{d}x = \frac{1}{2}\left(\sec x \tan x + \ln(\sec x + \tan x)\right) + C .$$ [8]

(b) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y \tan x = 2\sec^2 x, \ 0 \leq x < \frac{\pi}{2}, \text{ given that } y = 1 \text{ when } x = 0 .$$

(i) Find an integrating factor and hence solve the differential equation, giving your answer in the form $y = f(x)$.

(ii) Starting with the differential equation, show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 2\sec^2 x \tan x .$$

(iii) Hence, by using your calculator to draw two appropriate graphs or otherwise, find the x-coordinate of the point of inflection on the graph of $y = f(x)$. [16]

6. [Maximum mark: 14]

(a) **Figure 1**

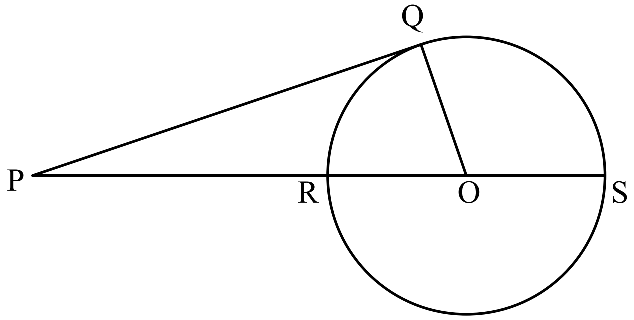


Figure 1 shows a tangent [PQ] at the point Q of a circle and a line [PS] meeting the circle at the points R, S and passing through the centre O of the circle.

(i) Show that $\text{PQ}^2 = \text{PR} \times \text{PS}$.

(ii) State briefly how this result can be generalized to give the tangent-secant theorem. [4]

(b) **Figure 2**

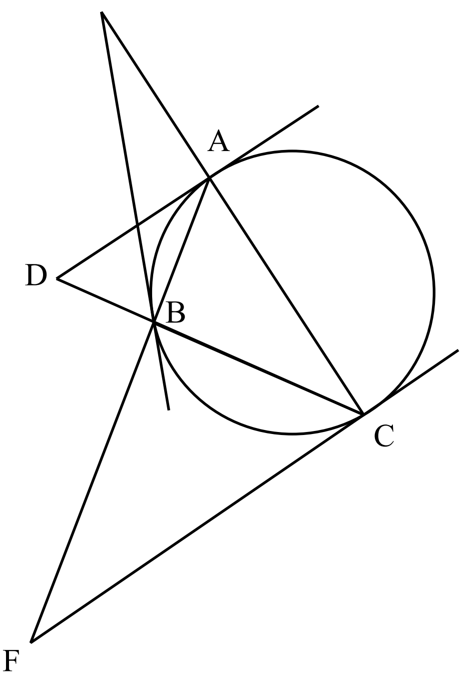


Figure 2 shows a triangle ABC inscribed in a circle. The tangents at the points A, B, C meet the opposite sides of the triangle externally at the points D, E, F respectively.

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(Question 6 continued)

(i) Show that $\frac{\text{AD}^2}{\text{BD}^2} = \frac{\text{CD}}{\text{BD}}$.

(ii) By considering a pair of similar triangles, show that

$$\frac{\text{AD}}{\text{BD}} = \frac{\text{AC}}{\text{AB}} \text{ and hence that } \frac{\text{CD}}{\text{BD}} = \frac{\text{AC}^2}{\text{AB}^2}.$$

(iii) By writing down and using two further similar expressions, show that the points D, E, F are collinear. [10]

7. [Maximum mark: 18]

The discrete random variable X has the following probability distribution.

$$\text{P}(X = x) = \frac{kx}{3^x} \text{ where } x \in \mathbb{Z}^+ \text{ and } k \text{ is a constant.}$$

(a) (i) Write down the first three terms of the infinite series for $G(t)$, the probability generating function for X.

(ii) Determine the radius of convergence of this infinite series.

(iii) By considering $\left(1 - \frac{t}{3}\right)G(t)$, show that

$$G(t) = \frac{3kt}{(3-t)^2}.$$

(iv) Hence show that $k = \frac{4}{3}$. [10]

(b) (i) Show that $\ln G(t) = \ln 4 + \ln t - 2\ln(3 - t)$.

(ii) By differentiating both sides of this equation, determine the values of $G'(1)$ and $G''(1)$.

(iii) Hence find $\text{Var}(X)$. [8]

8. [Maximum mark: 17]

The set $S_n = \{1, 2, 3, \ldots, n-2, n-1\}$, where n is a prime number greater than 2, and $\times_n$ denotes multiplication modulo n.

(a) (i) Show that there are no elements $a, b \in S_n$ such that $a \times_n b = 0$.

(ii) Show that, for $a, b, c \in S_n$, $a \times_n b = a \times_n c \Rightarrow b = c$. [4]

(b) Show that $G_n = \{S_n, \times_n\}$ is a group. You may assume that $\times_n$ is associative. [4]

(c) (i) Show that the order of the element $(n-1)$ is 2.

(ii) Show that the inverse of the element 2 is $\frac{1}{2}(n+1)$.

(iii) Explain why the inverse of the element 3 is $\frac{1}{3}(n+1)$ for some values of n but not for other values of n.

(iv) Determine the inverse of the element 3 in G_{11}.

(v) Determine the inverse of the element 3 in G_{31}. [9]

9. [Maximum mark: 17]

(a) The point (x, y) is rotated through an anticlockwise angle α about the origin to become the point (X, Y). Assume that the rotation can be represented by

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Show, by considering the images of the points $(1, 0)$ and $(0, 1)$ under this rotation that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$ [3]

(b) The hyperbola with equation $x^2 - 4xy - 2y^2 = 3$ is rotated through an acute anticlockwise angle α about the origin.

(i) By expressing (x, y) in terms of (X, Y), determine the equation of the rotated hyperbola in terms of X and Y.

(ii) Verify that the coefficient of XY in the equation is zero when $\tan\alpha = \frac{1}{2}$.

(iii) Determine the equation of the rotated hyperbola in this case, giving your answer in the form $\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1$.

(iv) Hence find the coordinates of the foci of the hyperbola prior to rotation. [14]